ELSEVIER

Available online at www.sciencedirect.com

SCIENCE DIRECT®

J. Math. Anal. Appl. 315 (2006) 154–166

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Dissipative control system for the stochastic nonlinear H^∞ problems ☆

Zhengmei Hou [a,b,*], Sanyang Liu [a], Yong Zhou [a,b]

[a] *Department of Applied Mathematics, Xidian University, Xi'an 710071, China*
[b] *Department of Stat. and Info., Institute of Xinjiang Economy and Finance, Urmqi 830012, China*

Received 12 March 2004

Available online 12 September 2005

Submitted by William F. Ames

Abstract

We characterize functions satisfying a dissipative inequality associated with a stochastic control problem. Such a characterization is provided in terms of an upper generalized Gaussian solution, or a viscosity supersolution to a partial differential equation called Hamilton–Jacobi equation (H–J). Links between upper generalized Gaussian solutions and viscosity supersolutions to Hamilton–Jacobi equation are studied. Finally it shows that generalized Gaussian solutions is identical to viscosity solutions to Hamilton–Jacobi equation.

Keywords: H–J equation; Stochastic nonlinear H^∞-control; Generalized Gaussian and viscosity solutions to PDEs

☆ This research is supported by National Science Foundation Grant No. 69972036.

* Corresponding author.
E-mail address: zyhzm2002@yahoo.com.cn (Z. Hou).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.07.035

1. Introduction

Consider a company whose net assets value is as follows:

$$X(t) = x + \int_0^t \mu\bigl(s, u(s)\bigr)\,ds + \int_0^t u(s)\sigma\,dW_s - D_t,$$

where, $X(0) = x$ is initial investment, given a filtered probability space (Ω, F, F_t, P) and one-dimensional standard Brownian motion W_t (with $W_0 = 0$) on it, adapted to the filtration F_t, μ_t is expected profit rate per unit time, $\{D_t\}$ is the total dividend distribution up to time t, u_s is a business policy, $\pi = (u_t, D_t; t \geqslant 0)$ is a control policy, we denote the set of all admissible controls policy by Π.

The performance function associated with each control π is

$$J_x(\pi) = E_x \int_0^\tau e^{-\gamma t}\,dD_t,$$

whereE_x denote the conditional expectation operator, $\gamma > 0$ is a priori given discount factor (used in calculating the present value of the future dividends). The objective is to find a control policy π to maximize the expected present value of the total dividend distributions

$$V(x) = \sup J_x(\pi) = \sup E_x \int_0^\tau e^{-\gamma t}\,dD_t$$

and meanwhile find a proper initial investment x such that $V(x)/\|x\|^2_{L^2[0,T]}$ is not less than some fixed constant γ^2, i.e., the expected present value of the total dividend distributions is the positive multiple of the initial investment x.

We now reduced the above problem to a more general one, consider the following control problem:

$$X_r^{t,x,u} = x + \int_t^r b\bigl(s, X_s^{t,x,u}, u(s)\bigr)\,ds + \int_t^r \sigma\bigl(s, X_s^{t,x,u}, u(s)\bigr)\,dW_s, \tag{1}$$

where $\{W_s;\ s \geqslant 0\}$ is a m-dimensional standard Brownian motion on a probability space (Ω, F, P) and $u(\cdot) \in U$ is a control. Here t is initial time, $X(t) = x$ is initial value.

Getting a control $u(\cdot)$ and a proper initial value x ensuring that the following so-called L^2-gain (gains to investment):

$$\frac{E\int_t^T f(s, X_s^{t,x,u}, u(s))\,ds}{\|x\|^2_{L^2[0,T]}}$$

is not less than some fixed constant γ^2 is one of the question addressed by H^∞ theory (in fact, the problem we consider is only a extended version of the classical deterministic problem, see [1] or [2]), where f is a measurable gain function.

A classical reformation of this problem consists in investigation of the value function of the optimal control problem described below.

The goal of the controller u is to maximize a performance function J_γ given by

$$J_\gamma(t,x,u) := E\int_t^T \left[f\left(s, X_s^{t,x,u}, u(s)\right) - \gamma^2\|x\|^2\right] ds \tag{2}$$

against all possible controller. The result of this optimal action of the controller is a quantity which depends on the initial conditions of the system (1):

$$V(t,x) = \sup_{u\in\overline{U}} J_\gamma(t,x,u), \tag{3}$$

here $\overline{U}$ is the set of U-valued progressively measurable stochastic process on $[0,T]$.

We prove that the value function $V(t,x)$ satisfy a partial differential inequality: the Hamilton–Jacobi–Bellman (H–J–B) inequality. The main problem consists in characterizing solutions of this partial differential inequality thanks to some monotonicity properties−dissipative inequality—of the performance function J_γ along suitable trajectories of the system.

When V is smooth—we emphasize that our study includes the non-smooth case—this equality can be written in the following form:

$$-V_t(t,x) + H\left(t,x,-V_x(t,x),-V_{x,x}(t,x)\right) = 0, \quad \forall (t,x)\in R\times R^d, \tag{3$'$}$$

where the subscripts denoted partial derivatives and H is the Hamiltonian for the problem and is defined as

$$H(t,x,p,P) := \inf_{u\in\overline{U}}\left\{\operatorname{trace}\left[P\cdot\frac{1}{2}\sigma(t,x,u)\sigma^T(t,x,u)\right] + p^T\cdot b(t,x,u) - \bar{f}(t,x,u)\right\},$$

where $\bar{f}(t,x,u) = f(t,x,u) - \gamma^2\|x\|^2$.

We recall the definition of dissipative inequality (see [3]) associated to some extended function $Y(\cdot,\cdot): R_+\times R^d \to R\cup\{\infty\}$. Consider a measurable function $u(\cdot)$. If, for any measurable function f, we have

$$EY\left(t_2, X_{t_2}^{t,x,u}\right) - EY\left(t_1, X_{t_1}^{t,x,u}\right) \leqslant E\int_{t_1}^{t_2}\left[\gamma^2\|x\|^2 - f\left(s, X_s^{t,x,u}, u(s)\right)\right] ds, \tag{4}$$

where $t_2 > t_1$, $X_s^{t,x,u}$ is given by (1), then the function Y is called a storage function.

The situation of smooth occurs rarely, thus the problem will be reduced to the statement of a criterion in terms of a PDE allowing us to determine storage functions. This is related to a paper of James [4] who proved—in the continuous and deterministic case—that storage functions are viscosity subsolution to some PDE and that any continuous viscosity subsolution is a storage function. In the present work, we prove—in the stochastic case—the equivalence between storage functions, generalized solutions and viscosity solutions to the H–J–B in the lower semi-continuous case. We briefly describe how the paper is organized. In Section 2 we introduce the upper generalized Gaussian derivative and some definitions associated with it; in the following section we study stochastic dissipative system and H–J–B equation; in the last section obtain the relation between storage functions, generalized solutions and viscosity solutions.

2. Preliminary

The standing assumptions are as follows:

1. b, σ are bounded, continuous on $[0, T] \times R^d \times U$ and Lipschitz continuous in (t, x) and uniformly in u, $\sigma(t, x, u)$ is a $d \times m$-dimensional matrix.
2. f is continuous, Lipschitz continuous in (t, x) and uniformly in u, and has at most polynomial growth in x.
3. U is a compact metric space.

The first-order generalized directional derivative of a function I at $x \in R^d$ in the direction $v \in R^d$, $I^\circ(x; v)$, is defined (at least if I is Lipschitz) by Clarke [5] as

$$I^\circ(x; v) := \limsup_{y \to x,\ h \to 0^+} \frac{I(y + vh) - I(y)}{h}.$$

The second-order generalized directional derivative at x has been defined by Cominetti and Correa [6] as a functional on $R^d \times R^d$, i.e.,

$$I^{\circ\circ}(x; u, v) := \limsup_{y \to x,\ s,\ t \to 0} \frac{I(y + tv + su) - I(y + tv) - I(y + su) + I(y)}{st}.$$

For stochastic control, however, where second-order H–J–B equations arise, it is more convenient to define a second-order derivative and a second-order differential somewhat differently. Let S^d be the symmetric $d \times d$-dimensional matrices and let P^d be the cone of nonnegative semi-definite elements of S^d. Let $\Re \in R^d$ be an open set and let $BM(\Re)$ be the set of locally bounded measurable real-valued functions defined on $\Re$. For a function $I \in BM(\Re)$ we provide a differential of I at x as a function on $P^d \times R^d$ as follows (see [7]).

Definition 2.1. For $a \in P^d$ and $b \in R^d$ the upper, respectively lower, generalized Gaussian derivative of I at x in the direction b with covariance a is

$$I^G(x; b, a) = \limsup_{(s,y) \to (t,x),\ h \to 0^+} \frac{E\{(\phi I)(y + bh + \theta W_h) - I(y)\}}{h}, \tag{5}$$

respectively

$$\begin{aligned} I_G(x; b, a) &= \liminf_{(s,y) \to (t,x),\ h \to 0^+} \frac{E\{(\phi I)(y + bh + \theta W_h) - I(y)\}}{h} \\ &= -(-I)^G(x; b, a). \end{aligned} \tag{5$'$}$$

We also provide the result as follows without proof (see [7]):

$$I^G(x;-b,-a) = (-I)^G(x;b,a) = -I_G(x;b,a), \tag{6}$$

where ϕ is any infinitely differential function of compact support that is equal to 1 in a neighborhood of x, where θ is any $d \times m$-dimensional matrix such that $\theta\theta^T = 2a$ and where W is a standard m-dimensional Brownian motion, E stands for expectation.

Remark 1. Our definition is somewhat different from that of [7], however, it makes no difference between them essentially. It can be shown that $I^G(x;b,a)$ and $I_G(x;b,a)$ are independent of the choice of ϕ, θ, W (see [7, Proposition 3.4]).

Definition 2.2 (see [8]). Let $Z \in C([t,T] \times R^d)$, the right subdifferential (respectively right superdifferential) of Z at $(t,x) \in R \times R^d$, denoted by $\partial_{t+,x}^{1,2,-} Z(t,x)$, respectively $\partial_{t+,x}^{1,2,+} Z(t,x)$, is a set defined by

$$\begin{aligned}
&\partial_{t+,x}^{1,2,-} Z(t,x) \\
&\quad = \Bigg\{(\beta_0,\beta,\alpha) \in R \times R^d \times S^d: \\
&\quad \liminf_{y\to x,\, h\to 0^+} \frac{Z(t+h,y) - Z(t,x) - \beta_0 h - \beta^T(y-x) - \frac{1}{2}(y-x)^T\alpha(y-x)}{h + \|y-x\|^2} \geqslant 0\Bigg\}
\end{aligned}$$

respectively

$$\partial_{t+,x}^{1,2,+} Z(t,x) = \Big\{(\beta_0,\beta,\alpha) \in R \times R^d \times S^d]dvt \limsup_{y\to x,\ h\to 0^+} \{\ldots\} \leqslant 0\Big\},$$

where $\partial_{t+,x}^{1,2,-} Z(t,x)$ denotes first subdifferential in $t+$, second subdifferential in x with respect to $Z(t,x)$.

Remark 2. To study stochastic control problems, many authors make use of the superdifferential $\partial_{t+,x}^{1,2,+} Z(t,x)$ and subdifferential $\partial_{t+,x}^{1,2,-} Z(t,x)$ obtained by replacing the right-side limit $h \to 0^+$ in the above definition by the two-side limit $h \to 0$ (e.g., [9–11]). The right-sided differentials has been studied extensively in [8] and proved to be more useful than the two-sided differential in treating stochastic control problem (see, e.g., [8, Remark 4.1] and [12]).

These definitions are related by a lemma.

Lemma 2.3. *Consider a function $\varphi : [0,T] \times R^d \to R \cup \{+\infty, -\infty\}$, then*

$$\begin{aligned}
\partial_{t+,x}^{1,2,-} \varphi(t,x) = \big\{&(\beta_0,\beta,\alpha) \in R \times R^d \times S^d: \\
&\varphi^G(t,x;b,a) \geqslant \beta_0 + \langle\beta,b\rangle + \langle\alpha,a\rangle\ \forall (a,b) \in S^d \times R^d\big\}, \\
\partial_{t+,x}^{1,2,+} \varphi(t,x) = \big\{&(\beta_0,\beta,\alpha) \in R \times R^d \times S^d: \\
&\varphi_G(t,x;b,a) \leqslant \beta_0 + \langle\beta,b\rangle + \langle\alpha,a\rangle\ \forall (a,b) \in S^d \times R^d\big\},
\end{aligned}$$

where $a = \frac{1}{2}\theta\theta^T$, we write $\langle e,f\rangle$ for $e^T \cdot f$ if $e, f \in R^d$ and for trace$(e \cdot f)$ *if $e, f \in S^d$.*

Proof. Consider $(\beta_0, \beta, \alpha) \in \partial^{1,2,-}_{t+,x}\varphi(t,x)$. Given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\begin{aligned}
&\varphi(s,y) - \varphi(t,y) - \beta_0(s-t) - \beta^T(y-x) - \frac{1}{2}(y-x)^T\alpha(y-x)\\
&\quad \geqslant -\varepsilon\big\{|s-t|^2 + \|y-x\|^2\big\} + (y-x)^T\alpha_1(s-t) + \frac{1}{2}\alpha_0(s-t)^2,
\end{aligned} \tag{7}$$

for $\|(s,y)-(t,x)\| \leqslant \delta$. For $a = \frac{1}{2}\theta\theta^T \in P^d$, $b \in R^d$, let $s = t+h$, $y = x + bh + \theta W_h$ for a suitable Brownian motion. If h is sufficiently small, then by Chebychev's inequality $\|(s,y)-(t,x)\| \leqslant \delta$ except on a set of probability measure less than kh^2, where k is a constant that may be depend on ε and a but not h. Let ϕ be a smooth function of compact support, equal to one on $\|(s,y)-(t,x)\| \leqslant \delta$. Since φ is bounded near (t,x), then from (7) it follows that

$$\begin{aligned}
&E\phi\varphi(t+h, x+bh+\theta W_h) - \varphi(t,x)\\
&\quad \geqslant \beta_0 h + \langle\beta, b\rangle h + \langle\alpha, a\rangle h - \varepsilon\big\{h^2 + \|b\|^2h^2 + 2\|a\|h\big\} + o(h^2),
\end{aligned}$$

and hence, after dividing by h, letting $h \to 0^+$ and $\varepsilon > 0$, i.e., pass to the upper limit, then

$$\varphi^G(t,x;b,a) \geqslant \beta_0 + \langle\beta, b\rangle + \langle\alpha, a\rangle.$$

When $(\beta_0, \beta, \alpha) \in \partial^{1,2,+}_{t+,x}\varphi(t,x)$, the proof is similar and is omitted. □

3. Stochastic dissipative system and H–J–B equation

Now let us consider the stochastic control system (1). In what follows we always assume that $x_2 = EX^{t,x,u}_{t_2}$, $x_1 = EX^{t,x,u}_{t_1}$, $x_n = EX^{t,x,u}_{t_n}$ and $X^{t,x,u}_s$ is given by (1).

Definition 3.1. The stochastic control system (1) is called dissipative system, if there exists a function $\eta : R_+ \times R^d \to R \cup \{\infty\}$ (called a storage function) such that for every $u \in L^2_{\mathrm{loc}}(R_+, U)$ and any solution $X^{t,x,u}_{\cdot}$ defined on some time interval $[T_1, T_2]$ we have

$$\eta(t_1, x_1) \geqslant \eta(t_2, x_2) + E\int_{t_1}^{t_2} \tilde{f}(t,x,u)\,ds, \quad \forall T_1 \leqslant t_1 \leqslant t_2 \leqslant T_2,$$

where η is a bounded measurable function, $\tilde{f}(t,x,u) = f(s, X^{t,x,u}_s, u(s)) - \gamma^2\|x\|^2$.

Inequality of Definition 3.1 is called the dissipative inequality. In general, the storage function is neither unique nor continuous. As we show semicontinuous storage functions can be studied as sub/supersolutions to a H–J–B equation. However, first we prove $V(t,x)$ is a storage function.

Proposition 3.1. *There exists a control u and a proper initial value x such that* (3) *holds if and only if $V(t,x)$ is a storage function, i.e., $V(t,x)$ satisfying* (4).

Proof. Consider $0 \leqslant t_1 \leqslant t_2 \leqslant T_2$. Denote by $\mho(t_1)$ the set of measurable controls on $[t_1, T_2]$ whose restriction to $[t_1, t_2]$ is equal to u, then

$$
\begin{aligned}
V(t_1,x_1) &= \sup_{\bar u\in\mho(t_1)} J_\gamma(t_1,x_1,\bar u)\\
&= \sup_{\bar u\in\mho(t_1)}\left\{E\int_{t_1}^{t_2}\left[f\left(s,X_s^{t,x,\bar u},\bar u(s)\right)-\gamma^2\|x\|^2\right]ds+J_\gamma(t_2,x_2,\bar u)\right\}\\
&\geqslant E\int_{t_1}^{t_2}\left[f\left(s,X_s^{t,x,u},u(s)\right)-\gamma^2\|x\|^2\right]ds+\sup_{\tilde u\in\mho(t_2)} J_\gamma(t_2,x_2,\tilde u).
\end{aligned}
$$

Hence

$$
V(t_1,x_1)\geqslant V(t_2,x_2)+E\int_{t_1}^{t_2}\tilde f(t,x,u)\,ds. \qquad \square
$$

In what follows, we prove that lower and upper envelopes of a storage function are again storage functions.

Proposition 3.2. *If V is a storage function, then so are its lower and upper envelopes V_* and V^*.*

Recall that the lower/upper envelope V_*, V^* of V is the largest lower/smallest upper semicontinuous function which is smaller/greater than V. In a shorter way, $\mathrm{Epi}(V_*)$ is the closure of the epigraph of V: $\mathrm{Epi}(V_*):=\mathrm{cl}(\mathrm{Epi}\,V)$. The upper envelope V^* is defined by considering the hypograph: $\mathrm{Hypo}\,V^*:=\mathrm{cl}(\mathrm{Hypo}\,V)$.

Proof. Fix $u\in L^2_{\mathrm{loc}}$, let $t_1\leqslant t_2$. By the very definition of V_*, there exists a sequence $(t_n,x_n)_n$ converging to (t_1,x_1) such that

$$
\liminf_{(t,y)\to(t_1,x_1)} V(t,y)=\liminf_{n\to\infty} V(t_n,x_n)=V_*(t_1,x_1).
$$

We consider a sequence $x_n(\cdot)$ corresponding to the fixed $u(\cdot)$ such that $x_n(t_n)=x_n$ for any $n\geqslant 0$. Since V is a storage function one gets, for any t,

$$
V(t_n,x_n)\geqslant V\left(t,x_n(t)\right)+E\int_{t_n}^{t}\left[f\left(s,X_n(s),u(s)\right)-\gamma^2\|x\|^2\right]ds.
$$

By Gronwall's lemma and (1), $\|X_n(s)-X(s)\|_{\mathrm{rv}}$ ($\|\cdot\|_{\mathrm{rv}}$ denotes the norm of random variable) converges uniformly to 0 on $[t_1,t]$. So passing to the lower limit in the previous inequality we obtain

$$
V_*(t_1,x_1)\geqslant \liminf_{n\to\infty}\left\{V\left(t,x_n(t)\right)+E\int_{t_n}^{t}\left[f\left(s,X_n(s),u(s)\right)-\gamma^2\|x\|^2\right]ds\right\}.
$$

However, $\liminf_{n\to\infty} V(t, x_n(t)) \geqslant \liminf_{(s,y)\to(t,x(t))} V(s, y) =: V_*(t, x(t))$, and since f is continuous we can deduce from Lebesgue's theorem and from the standing assumption 2 that

$$V_*(t_1, x_1) \geqslant V_*\big(t, x(t)\big) + E\int_{t_1}^{t} \big[f\big(s, X_1(s), u(s)\big) - \gamma^2\|x\|^2\big]\,ds,$$

for any t. By taking $t = t_2$ yields

$$V_*(t_1, x_1) \geqslant V_*\big(t_2, x(t_2)\big) + E\int_{t_1}^{t_2} \big[f\big(s, X_1(s), u(s)\big) - \gamma^2\|x\|^2\big]\,ds. \qquad \square$$

The proof for the upper envelope V^* is very similar and is omitted. Because of the above result, we only study lower semicontinuous storage functions.

Theorem 3.3. *Assume that the standing assumptions* 1, 2 *hold true. If* $V \in W^{1,2,\infty}_{\mathrm{loc}}$ ($W^{m,p}$ *denotes Sobolev space*) *is a storage function, then, for all* $(t, x) \in \mathrm{Dom}(V)$,

$$\begin{cases} \sup_{u\in U} V^G(t, x; b, a) + \bar f(t, x, u) \leqslant 0, \\ \sup_{u\in U} (-V)^G(t, x; -b, -a) + \bar f(t, x, u) \leqslant 0, \end{cases}$$

where $b = b(t, x, u)$, $\sigma = \sigma(t, x, u)$, $a = \frac{1}{2}\sigma\sigma^T$, $\bar f(t, x, u) = f(t, x, u) - \gamma^2\|x\|^2$.

Proof. Firstly, we prove the first inequality. Consider the system (1), we have

$$X_r^{t,x,u} = x + \int_0^r b\big(t + s, X_s^{t,x,u}, u(s)\big)\,ds + \int_0^r \sigma\big(t + s, X_s^{t,x,u}, u(s)\big)\,dW_s, \tag{8}$$

with

$$J_{t,x}(u) = E\int_0^{T-t} \tilde f(t, x, u)\,ds.$$

The corresponding value function is

$$V(t, x) = \sup_{u\in\overline{U}} J_{t,x}(u), \tag{9}$$

where $\overline{U}$ is the set of U-valued progressively measurable stochastic process on $[0, T]$.

When value function $V(t, x) = \inf_{u\in\overline{U}} J_{t,x}(u)$, it follows from a generalized dynamic programming argument [13, Propositions 5.9 and 5.11] that

$$\tilde\Gamma_r^{t,x,u} := \int_0^r \tilde f(t, x, u)\,ds + V\big(t + r, X_r^{t,x,u}\big)$$

is a submartingale for any $u \in \overline{U}$ and is a martingale if and only if u is optimal.

When value function is (9), we can easily obtain by using the same methods of [13, Propositions 5.9 and 5.11] that

$$\Gamma_r^{t,x,u} := \int_0^r \tilde{f}(t,x,u)\,ds + V\bigl(t+r, X_r^{t,x,u}\bigr) \tag{10}$$

is a supermartingale for any $u \in \overline{U}$ and is a martingale if and only if u is optimal. Hence we have for any $u \in \overline{U}$

$$E\bigl(\Gamma_r^{t,x,u} \mid F_0\bigr) \leqslant \Gamma_0^{t,x,u}, \tag{11}$$

namely

$$EV\bigl(t+r, X_r^{t,x,u}\bigr) \leqslant V(t,x) - E\int_0^r \tilde{f}(t,x,u)\,ds. \tag{12}$$

Let us take $u(\cdot)$ and u constant in (8), for $h > 0$, then

$$\begin{aligned} X_h^{s,y,u} &= y + \int_0^h b\bigl(s+r, X_r^{s,y,u}, u\bigr)\,ds + \int_0^h \sigma\bigl(s+r, X_r^{s,y,u}, u\bigr)\,dW_r \\ &= y + hb(t,x,u) + \sigma(t,x,u)W_h + \rho_1 + \rho_2, \end{aligned}$$

where

$$\begin{cases} \rho_1 = \int_0^h [b(s+r, X_r^{s,y,u}, u) - b(t,x,u)]\,ds, \\ \rho_2 = \int_0^h [\sigma(s+r, X_r^{s,y,u}, u) - \sigma(t,x,u)]\,dW_r. \end{cases}$$

Observe that the Lipschitz continuity in x of b and standard estimation show that

$$h^{-1}\rho_1 = g(s,y), \tag{13}$$

where $g(s,y)$ is a random function such that

$$\lim_{(s,\ y)\to(t,\ x),\ h\to 0^+} E\bigl|g(s,y)\bigr|^2 = 0.$$

Similarly

$$h^{-1/2}\rho_2 = g(s,y). \tag{14}$$

For the sake of clarity, we write $b(t,x,u)$, $\sigma(t,x,u)$ as b, σ, respectively, in the following proof, now,

$$\begin{aligned} V\bigl(s+h, X_h^{s,y,u}\bigr) - V(s,y) &= \bigl[V(s+h, y+hb+\sigma W_h + \rho_1 + \rho_2) \\ &\quad - V(s+h, y+hb+\sigma W_h)\bigr] \\ &\quad + \bigl[V(s+h, y+hb+\sigma W_h) - V(s,y)\bigr] \\ &= \Delta V + \bigl[V(s+h, y+hb+\sigma W_h) - V(s,y)\bigr]. \end{aligned} \tag{15}$$

But V_x is locally Lipschitz-continuous, so

$$\begin{aligned}\Delta V &= \big[V_x\big(s+h, y+hb+\sigma W_h+\xi(\rho_1+\rho_2)\big)\big](\rho_1+\rho_2)\\ &= V_x(s+h, y)(\rho_1+\rho_2)+\big[V_x\big(s+h, y+hb+\sigma W_h+\xi(\rho_1+\rho_2)\big)\\ &\quad - V_x(s+h, y)\big](\rho_1+\rho_2),\end{aligned}$$

where ξ is a random variable assuming values in $(0, 1)$. Set

$$\Delta_1 V = V_x\big(s+h, y+hb+\sigma W_h+\xi(\rho_1+\rho_2)\big) - V_x(s+h, y).$$

Then

$$\Delta_1 V \leqslant K\big|hb+\sigma W_h+\xi(\rho_1+\rho_2)\big| \leqslant K_1\big\{|h|+|W_h|+|\rho_1|+|\rho_2|\big\}$$

and hence

$$\lim_{(s,\ y)\to(t,\ x),\ h\to 0^+} h^{-1}E\Delta V = 0. \tag{16}$$

If now we set $a = \frac{1}{2}\sigma\sigma^T$, then it follows from (5), (12), (15) and (16) that

$$\begin{aligned}V^G(t,x;b,a) &= \limsup_{(s,y)\to(t,x),\ h\to 0^+} h^{-1}E\big\{\phi V\big(s+h, X_h^{s,y,u}\big) - V(s,y)\big\}\\ &= \limsup_{(s,y)\to(t,x),\ h\to 0^+} h^{-1}E\big\{V\big(s+h, X_h^{s,y,u}\big) - V(s,y)\big\}\\ &= \limsup_{(s,y)\to(t,x),\ h\to 0^+} h^{-1}E\big\{V(s+h, y+hb+\sigma W_h) - V(s,y)\big\}\\ &\leqslant - \limsup_{(s,y)\to(t,x),\ h\to 0^+} h^{-1}E\int_0^h \tilde f(s,x,u)\,ds\\ &= -\bar f(t,x,u)\end{aligned}$$

(the ϕ can be omitted since V has at most polynomial growth). Since u is arbitrary, we obtain the first inequality. The proof of the second inequality is very similar and is omitted. □

4. Storage function, generalized Gaussian solution, viscosity solution

If V is smooth, this implies that it is the unique classical solution of the H–J–B Eq. (3′), where the associated Hamiltonian is defined on $[0,T]\times R^d\times R^d\times S^d$. This situation occurs rarely, but it is true that V is always a weak (in the sense of distribution) solution, although the latter are not unique. We shall show that V satisfies the H–J–B Eq. (18) in a viscosity sense, at least if V possesses some regularity. Hence we suppose that $V\in W_{\mathrm{loc}}^{1,2,\infty}(\Re)$ so that $\frac{\partial V}{\partial x}$ and $\frac{\partial^2 V}{\partial x^2}$ exist almost everywhere (here $W^{m,p}$ denotes Sobolev space). We now define the Hamiltonian of the reformed H^∞ control problem (2) and (3):

$$H(t,x,p,P) := \inf_{u\in\overline{U}}\big\{\big\langle P, a(t,x,u)\big\rangle + \big\langle p, b(t,x,u)\big\rangle - \bar f(t,x,u)\big\} \tag{17}$$

(the interpretation of $\langle\cdot,\cdot\rangle$ is identical to that of Lemma 2.1), where $a = \frac{1}{2}\sigma\sigma^T$. We write $b(t,x,u)$, $\sigma(t,x,u)$ as b, σ, respectively, in the following and no more special claim.

Consider the associated H–J–B equation:

$$-\frac{\partial \Lambda}{\partial t}(t,x) + H\left(t, x, -\frac{\partial \Lambda}{\partial x}(t,x), -\frac{\partial^2 \Lambda}{\partial x^2}(t,x)\right) = 0. \tag{18}$$

Definition 4.1. A function $\Theta \in W^{1,2,\infty}_{\mathrm{loc}}([0,T] \times R^d)$ is an upper generalized Gaussian solution to (18) if and only if, for all $(t,x) \in \mathrm{Dom}(\Theta)$

$$\sup_{u \in \overline{U}} \Theta^G(t, x : b, a) + \bar{f}(t,x,u) \leqslant 0.$$

A function $V \in W^{1,2,\infty}_{\mathrm{loc}}([0,T] \times R^d)$ is a lower generalized Gaussian solution to (18) if and only if, for all $(t,x) \in \mathrm{Dom}(\Theta)$

$$\sup_{u \in \overline{U}} \Theta_G(t, x : b, a) + \bar{f}(t,x,u) \geqslant 0.$$

A function Θ is a generalized Gaussian solution to (18) if and only if it is both an upper and a lower generalized Gaussian solution to (18).

Proposition 4.2. *Assume that the standing assumptions* 1, 2 *hold true. If V is a storage function, then V is a generalized Gaussian solution to* (18).

Proof. By Propositions 3.1, Theorem 3.3, and Definition 4.1 it follows promptly that V is a generalized Gaussian solution to (18). □

Now, we recall the definition of viscosity introduced in [14,15] in the continuous case.

Definition 4.3. Consider a Hamiltonian $H : [0,T] \times R^d \times R^d \times S^d \to R$. Then a function $\Theta : [0,T] \times R^d \to R \cup \{+\infty, -\infty\}$ is a viscosity supersolution to the PDE

$$-\frac{\partial \Lambda}{\partial t}(t,x) + H\left(t, x, -\frac{\partial \Lambda}{\partial x}(t,x), -\frac{\partial^2 \Lambda}{\partial x^2}(t,x)\right) = 0$$

if and only if

$$\forall (t,x) \in \mathrm{Dom}(\Theta),\ \forall (\beta_0, \beta, \alpha) \in \partial^{1,2,-}_{t+,x} \varphi(t,x), \quad -\beta_0 + H(t,x,-\beta,-\alpha) \geqslant 0;$$

a function Θ is a viscosity subsolution to the PDE if and only if

$$\forall (t,x) \in \mathrm{Dom}(\Theta),\ \forall (\beta_0, \beta, \alpha) \in \partial^{1,2,+}_{t+,x} \varphi(t,x), \quad -\beta_0 + H(t,x,-\beta,-\alpha) \leqslant 0;$$

a function Θ is a viscosity solution if it is a supersolution and a subsolution.

Now, we check, like [4], that any storage function is a viscosity solution. First we show a result as follows.

Proposition 4.4. *Consider a function $V \in W^{1,2,\infty}_{\mathrm{loc}}(\Re)$.*

- *If $V(\cdot,\cdot)$ is an upper generalized Gaussian solution to* (18), *then it is a viscosity superstition to* (18);

- *If $V(\cdot,\cdot)$ is an lower generalized Gaussian solution to* (18), *then it is a viscosity subsolution to* (18);
- *If $V(\cdot,\cdot)$ is a generalized Gaussian solution to* (18), *then it is a viscosity solution to* (18).

Proof. Let $V(\cdot,\cdot)$ be an upper generalized Gaussian solution to (18), then

$$\sup_{u\in U} V^G(t,x:b,a)+\bar{f}(t,x,u)\leqslant 0. \tag{19}$$

Consider $(\beta_0,\beta,\alpha)\in\partial^{1,2,-}_{t+,x}V(t,x)$. Thanks to Lemma 2.3, we obtain

$$V^G(t,x;b,a)\geqslant\beta_0+\langle\beta,b\rangle+\langle\alpha,a\rangle. \tag{20}$$

By adding $\bar{f}(t,x,u)=f(t,x,u(\cdot))-\gamma^2\|x\|^2$ to both sides of inequality (20) and by taking the 'sup', we obtain (where $a=a(t,x,u(\cdot)$, $b=b(t,x,u(\cdot))$,

$$0\geqslant\sup_{u\in\overline{U}} V^G(t,x;b,a)+\bar{f}(t,x,u)\geqslant\sup_{u\in\overline{U}}\bigl\{\beta_0+\langle\beta,b\rangle+\langle\alpha,a\rangle+\bar{f}(t,x,u)\bigr\}. \tag{21}$$

Hence from (17) and (21) we obtain

$$-\beta_0+H(t,x,-\beta,-\alpha)\geqslant 0.$$

By Definition 4.3, V is a viscosity supersolution to (18). The proof of the second statement is very similar and is omitted. And the third statement is obvious. □

By Propositions 4.2 and 4.4 we obtain that any storage function is a viscosity solution.

References

[1] T. Basar, P. Bernhard, H^∞-Optimal Control and Minimax Design Problems, Birkhäuser, Boston, MA, 1991.
[2] H. Knobloch, D. Flockerzi, Topics in Control Theory, Birkhäuser, Basel, 1995.
[3] J.C. Willems, Dissipative dynamical systems, Arch. Ration. Mech. Anal. 45 (1972) 321–351.
[4] M.R. James, A partial differential inequality for dissipative nonlinear system, Systems Control Lett. 21 (1993) 315–320.
[5] F.H. Clarke, Optimization and Nonsmooth Analysis, Wiley, New York, 1983.
[6] R. Cominetti, R. Correa, A generalized second-order derivative in nonsmooth optimization, SIAM J. Control Optim. 28 (1990) 789–809.
[7] U.G. Haussmann, A probabilistic approach to the generalized Hessian, Math. Oper. Res. 17 (1992) 411–443.
[8] X.Y. Zhou, A unified treatment of maximum principle and dynamic programming in stochastic controls, Stoch. Stoch. Rep. 36 (1991) 137–161.
[9] M.G. Crandall, H. Ishii, P.L. Lions, A user's guide to viscosity solutions, Bull. Amer. Math. Soc. 27 (1992) 1–67.
[10] P.L. Lions, Optimal control of diffusion process and Hamilton–Jacobi–Bellman equations I, Comm. Partial Differential Equations 8 (1983) 1101–1134.
[11] P.L. Lions, Optimal control of diffusion process and Hamilton–Jacobi–Bellman equations II: viscosity solutions and uniqueness, Comm. Partial Differential Equations 11 (1983) 1229–1276.
[12] U.G. Haussmann, Generalized solutions of the Hamilton–Jacobi equation of stochastic control, SIAM J. Control. Optim. 32 (1994) 728–743.
[13] U.G. Haussmann, J.-P. Lepeltier, On the existence of optimal controls, SIAM J. Control Optim. 28 (1990) 851–902.

[14] M.G. Crandall, P.L. Lions, Viscosity solutions of Hamilton–Jacobi equations, Trans. Amer. Math. Soc. 277 (1983) 1–42.

[15] M.G. Crandall, L.C. Evans, P.L. Lions, Some properties of viscosity solutions of Hamilton–Jacobi equations, Trans. Amer. Math. Soc. 282 (1984) 487–502.